A Petrov-Galerkin method for convection-dominated problems

B. Delsaute and F. Dupret
Centre for Systems Engineering and Applied Mechanics,
Université catholique de Louvain,
4 Avenue Georges Lemaître,
B-1348 Louvain-la-Neuve, Belgium
Phone: +32-10-472.365, Fax: +32-10-474.722, E-mail: delsaute@mema.ucl.ac.be

Abstract - The solution of the advection-diffusion problem represents a very important issue in numerical modeling in view of the large number of applications concerned. In the present paper, we develop an exact Finite Element Petrov-Galerkin method in order to solve this problem. Our approach is based on searching the test functions that provide exact nodal values for a selected class of solutions. Using these test functions in the general case induces a stabilizing upwinding effect which removes the wiggles obtained with the pure Galerkin method. The different issues to be addressed in order to build from this approach a general, robust, reliable, and accurate solution technique are discussed.

I. Introduction

A wide class of problems of fluid mechanics and heat and mass transfer is governed by the combined effects of transport and diffusion. Among the various numerical techniques available to solve these problems, the Galerkin Finite Element (FE) method should be the first mentioned in view of its simplicity and ease of implementation. In this case the discrete system is obtained from the continuous weak formulation by selecting the same finite dimensional subspaces for the shape functions (to discretize the unknowns) and the test functions (to discretize the equations).

It is well known that the Galerkin FE method is very well suited for diffusion-dominated problems while it performs quite badly when transport effects prevail. Therefore the solution of the advection-diffusion equation has been the object of extensive investigations in the literature [HUG 79], [BRO 82], [HUG 89], [BAI 93], [COC 00] since this generic equation exhibits the principal numerical difficulties to be addressed. A single parameter, the Péclet number Pe, which can be understood as the ratio of transport versus diffusion effects, governs the advection-diffusion equation. At low values of Pe, when diffusion dominates, the discrete eigenvalues of the stiffness (or Jacobian) matrix are basically real and the system is close to a functional minimization problem. Accordingly, the solution wiggles tend naturally to be strongly reduced (in the "energy norm"). On the contrary, at high values of Pe, when transport prevails, the discrete eigenvalues are closer and closer to being purely imaginary and the discrete system is no longer close to a minimization but better to a saddle point problem, without natural wiggle reduction effect. Several modifications of the Galerkin FE method have been considered in order to remove the solution oscillations at high Pe. In view of the rather poor accuracy of the Streamline Upwind (SU) technique [HUG 79], Brooks and Hughes [BRO 82] further developed the celebrated Streamline Upwind/Petrov-Galerkin (SUPG) technique, which was further justified in [JOH 84]. Later, in addition, the Galerkin/Least-Squares method [HUG 89], the Bubble methods [BAI 93] and the Discontinuous Galerkin methods [COC 00] were proposed to accurately solve the advection-diffusion problem without requiring such highly refined meshes as the Galerkin method does. All these methods, which basically result in moving to the left in the complex plane the dominant system eigenvalues, are consistent, since the exact solution is a solution of the discrete problem in the absence of geometrical boundary effects, while these methods are not conformal since the discrete system is not exactly obtained by introducing appropriate finite dimensional shape and test function spaces into the original continuous weak formulation.

Nevertheless, defining a both consistent and conformal FE method such as Galerkin in order to solve the advection-diffusion problem represents a highly attractive objective as being quite easy (i) to analyze and (ii) to implement. In this paper, consistent and conformal methods will be called exact Petrov-Galerkin FE methods, as obtained from the continuous weak formulation by selecting different finite dimensional subspaces for the shape and test functions. Our objective is to revisit the construction of exact Petrov-Galerkin FE methods. Restricting ourselves to the 1D and 2D steady problems, it will be shown that no definite conclusions against these methods can be drawn since previous investigations in the literature were performed too early, without benefitting from today's available research input.

II. Principles of the method

Starting from the 1D steady problem, an exact Petrov-Galerkin FE method is developed and extended to the 2D case.

A. One-dimensional case

To introduce the principles governing our method, let us consider the non-dimensional 1D steady advection-diffusion equation over the domain $\Omega =]0,1[$:

$$\frac{dT}{dx} - \frac{1}{\text{Pe}}\frac{d^2T}{dx^2} = 0 \tag{1}$$

where T is the temperature field and $\text{Pe} = \frac{vL}{\kappa}$ is the Péclet number (where v and κ are the assumed constant fluid velocity and thermal diffusivity while L is the domain length). With the boundary conditions $T(0) = T_0$ and $T(1) = T_1$, the exact solution of this boundary value problem is

$$T(x) = T_0 + (T_1 - T_0)\frac{e^{\text{Pe}x} - 1}{e^{\text{Pe}} - 1}.$$

A.1 Weak formulation

The classical weak formulation of the problem is:
Find $T \in S$ such that

$$\int_\Omega T'\frac{dT}{dx} + \frac{1}{\text{Pe}}\frac{dT'}{dx}\frac{dT}{dx}dx = \int_\Omega T' f dx \quad \forall T' \in V \tag{2}$$

where S is the affine manifold

$$S = \{T \in H^1(\Omega) | T(0) = T_0; T(1) = T_1\}$$

while V is the test function space

$$V = \{T \in H^1(\Omega) | T(0) = 0; T(1) = 0\} = H_0^1(\Omega)$$

The discrete problem writes as :
Find $T^h \in S^h$ such that

$$\int_\Omega T'^h\frac{dT^h}{dx}dx + \frac{1}{\text{Pe}}\frac{dT'^h}{dx}\frac{dT^h}{dx}dx = \int_\Omega T'^h f dx \qquad \forall T'^h \in V^h \tag{3}$$

where S^h is a finite dimensional submanifold of S and V^h a finite dimensional subspace of V to be defined from a partition of Ω. The latter consists of N subintervals $[x_{i-1}, x_i]$, $i = 1, \ldots N$, with $0 = x_0 < x_i < x_N = 1$.
Any function T^h of S^h has a unique expansion $T^h = \sum_{i=0}^N T_i^h \phi_i$ where the coefficient T_i^h is the value of T^h at node i, $T_i^h = T^h(x_i)$. In the same way, the functions defining V^h can be written in the form $T'^h = \sum_{i=0}^N T_i'^h \psi_i$, where the discrete test functions ψ_i remain to be defined.

A.2 Galerkin finite element approximation

The Galerkin method consists in considering the discrete weak formulation (3) with the discrete affine space S^h defined from V^h as underlying vector space. Hence as a basis for V^h we use the set of Lagrange basis functions

$$L = \{\phi_i | i = 0, \ldots, N\}$$

where ϕ_i is related to node x_i and is defined by $\phi_i \in V^h$ and $\phi_i(x_j) = \delta_{ij}$. The Lagrange functions exhibit the completion property, which means that constant functions belong to V^h and therefore that the method is globally and locally conservative [HUG 00].
The subspaces S^h and V^h are thus composed of identical collections of functions up to the affine part of S^h where the prescribed essential boundary conditions are introduced. As it is well-known, this approach fails when applied to convection-dominated flows while it satisfies the best approximation property for positive-definite symmetric problems.

A.3 Petrov-Galerkin finite element approximation

The Petrov-Galerkin approximation consists in selecting test functions ψ_i differing from the shape functions ϕ_i.

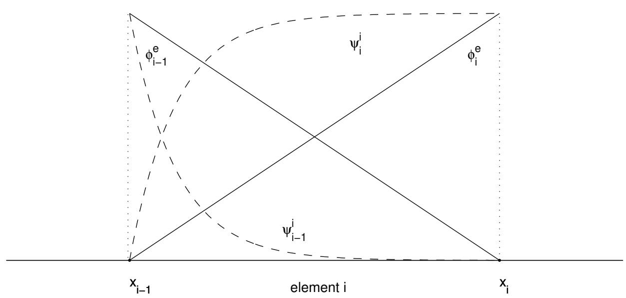

Fig. 1. Local linear shape functions and linear or upwinded test functions for element i.

The basic idea of our approach [PER 96], [NES 03] is to select test functions ψ_i that provide a nodally exact solution

$$T_i = T(x_i)$$

while preserving the completion property

$$\psi_i + \psi_{i+1} = 1 \quad \text{over } [x_{i-1}, x_i]$$

For the sake of clarity, test functions will be considered as perturbed shape functions of the form

$$\psi_i = \phi_i + \tilde{\phi}_i$$

where the perturbation function $\tilde{\phi}_i$ vanishes on element boundaries without any loss of generality.
Defining the 0-th moments of the perturbation function as

$$\mu_{0i}^i = \frac{1}{x_i - x_{i-1}}\int_{x_{i-1}}^{x_i} \tilde{\phi}_i^i dx$$

$$\mu_{0i-1}^i = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \tilde{\phi}_{i-1}^i dx$$

the obtention of a nodally exact solution only requires that

$$\mu_{0i}^i = -\frac{1}{\mathrm{Pe}_i} + \frac{e^{-\mathrm{Pe}_i} - e^{\mathrm{Pe}_i}}{2(1 - e^{-\mathrm{Pe}_i} - e^{\mathrm{Pe}_i})}$$

$$\mu_{0i-1}^i = \frac{1}{\mathrm{Pe}_i} + \frac{e^{\mathrm{Pe}_i} - e^{-\mathrm{Pe}_i}}{2(1 - e^{-\mathrm{Pe}_i} - e^{-\mathrm{Pe}_i})}$$

where $\mathrm{Pe}_i = \frac{vh_i}{\kappa} = \mathrm{Pe}\, h_i$ is the local Péclet number on element i. It should be noted that these conditions on μ_{0i}^i and μ_{0i-1}^i constrain but do not completely provide the perturbation functions $\tilde{\phi}_i^i$ and $\tilde{\phi}_{i-1}^i$. The latter can be selected in such way that the completion property is satisfied. This is important in order to ensure local and global energy conservation. An example of local test functions is shown in Fig. 1.

A simple asymptotic analysis shows that, for convection-dominated problems, these test functions provide an upwinding effect,

$$\lim_{\mathrm{Pe}_i \to \infty} \mu_{0i}^i = \frac{1}{2}$$

while for diffusion-dominated problems they are close to the shape functions,

$$\lim_{\mathrm{Pe}_i \to 0} \mu_{0i}^i = 0$$

Let us emphasize that our approach provides an exact Petrov-Galerkin FE method as opposed to the classical SUPG method [BRO 82], [JOH 84], which achieves stabilization by adding an upwinding term to the original weak formulation (3).

B. Two-dimensional case

Let us now try to extend the method to the non-dimensional 2D advection-diffusion equation. The boundary value problem consists in finding a temperature field $T(\mathbf{x})$ such that

$$\mathbf{v} \cdot \boldsymbol{\nabla} T - \frac{1}{\mathrm{Pe}} \Delta T = 0 \quad \text{in } \Omega \tag{4}$$

with $T = \overline{T} \quad$ on Γ and where Ω is an open bounded set of $\mathbb{R}^2$ with a piecewise linear boundary Γ, while $\overline{T} : \Gamma \to \mathbb{R}$ is prescribed and κ and $\mathbf{v}$ are constant thermal diffusivity and normalized fluid velocity.

Equation (4) admits the following 1D solutions:

$$T(\mathbf{x}) = 1$$

$$T(\mathbf{x}) = \mathbf{v}^{\perp} \cdot \mathbf{x}$$

$$T(\mathbf{x}) = e^{\frac{(\mathbf{m}\cdot\mathbf{v})(\mathbf{m}\cdot\mathbf{x})}{\kappa \mathbf{m}\cdot\mathbf{m}}}$$

where $\mathbf{v}^{\perp}$ is obtained from a quarter of turn of $\mathbf{v}$ and $\mathbf{m}$ is an arbitrary vector.

B.1 Weak formulation

The variational formulation of the boundary value problem (4) writes as:
Find $T \in S$ such that

$$\int_\Omega T' \mathbf{v} \cdot \boldsymbol{\nabla} T d\Omega + \frac{1}{\mathrm{Pe}} \int_\Omega \boldsymbol{\nabla} T' \cdot \boldsymbol{\nabla} T d\Omega = 0 \quad \forall T' \in V \tag{5}$$

with the function spaces

$$S = \{T \in H^1(\Omega) | T = \overline{T} \,\text{on}\, \Gamma\}$$

$$V = \{T \in H^1(\Omega) | T = 0 \,\text{on}\, \Gamma\} = H_0^1(\Omega)$$

B.2 Galerkin finite element approximation

The Galerkin finite element approximation corresponding to the previous weak formulation writes as:
Find $T^h \in S^h$ such that

$$\int_\Omega T'^h \mathbf{v} \cdot \boldsymbol{\nabla} T^h d\Omega + \kappa \int_\Omega \boldsymbol{\nabla} T'^h \cdot \boldsymbol{\nabla} T^h d\Omega = 0 \quad \forall T'^h \in V^h$$

where the finite element spaces $S^h \subset S$ and $V^h \subset V$ consist of continuous piecewise linear polynomials defined on a conforming triangulation $\mathcal{T}^h$ of Ω.

The Lagrange shape function ϕ_i associated to node i is defined by $\phi_i(\mathbf{x_j}) = \delta_{\mathbf{ij}}$ while its support extends over all the elements sharing this node (Fig. 2). It is convenient to define the local shape function ϕ_i^e as the restriction of ϕ_i over the element Ω^e, with $\phi_i|_{\Omega^e} = \phi_i^e$. For the sake of convenience, we adopt the local node numbering depicted on Fig. 3. The relationship between local and global numberings is provided by the integer expressions n_k^e $(k = 1, 2, 3)$. For example, in Fig. 3, $n_1^e = j, n_2^e = k, n_3^e = i$.

B.3 Petrov-Galerkin finite element approximation

We want to find test functions ψ_i that provide exact nodal values for the solution

$$Ae^{\frac{(\mathbf{m}\cdot\mathbf{v})(\mathbf{m}\cdot\mathbf{x})}{\kappa \mathbf{m}\cdot\mathbf{m}}} + B\mathbf{v}^{\perp} \cdot \mathbf{x} + C$$

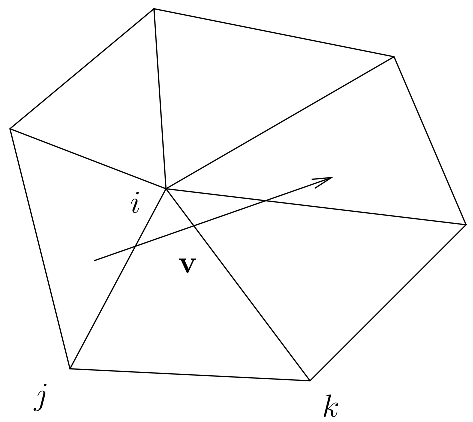

Fig. 2. Support of the shape and test functions associated to node i.

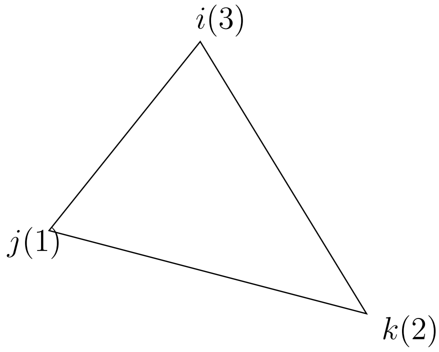

Fig. 3. Local numbering system.

Again, the test functions ψ_i are considered as modifications of the Lagrange shape functions,

$$\psi_i = \phi_i + \tilde{\phi}_i$$

The perturbation function $\tilde{\phi}_i$ is required to have the same support as ϕ_i and to vanish on the element boundaries. Moreover the set of test functions must contain the constant functions in order to ensure global and local conservation properties. Obtaining a nodally exact solution requires the following conditions to be satisfied by the 0-th order moments of the perturbations:

$$\begin{aligned}\mu^e_{01} = \Big[&(\mathrm{Pe}^e_1 - G^e_{11}) e^{-4G^e_{32}\mathrm{Pe}^{e,\mathbf{m}}_1} - G^e_{12} e^{-4G^e_{13}\mathrm{Pe}^{e,\mathbf{m}}_2} \\ &- G^e_{13} e^{-4G^e_{21}\mathrm{Pe}^{e,\mathbf{m}}_3}\Big]/D - \frac{1}{3}\end{aligned} \tag{6}$$

$$\begin{aligned}\mu^e_{02} = \Big[&(\mathrm{Pe}^e_2 - G^e_{22}) e^{-4G^e_{13}\mathrm{Pe}^{e,\mathbf{m}}_2} - G^e_{23} e^{-4G^e_{21}\mathrm{Pe}^{e,\mathbf{m}}_3} \\ &- G^e_{21} e^{-4G^e_{32}\mathrm{Pe}^{e,\mathbf{m}}_1}\Big]/D - \frac{1}{3}\end{aligned} \tag{7}$$

$$\begin{aligned}\mu^e_{03} = \Big[&(\mathrm{Pe}^e_3 - G^e_{33}) e^{-4G^e_{21}\mathrm{Pe}^{e,\mathbf{m}}_3} - G^e_{31} e^{-4G^e_{32}\mathrm{Pe}^{e,\mathbf{m}}_1} \\ &- G^e_{32} e^{-4G^e_{13}\mathrm{Pe}^{e,\mathbf{m}}_2}\Big]/D - \frac{1}{3}\end{aligned} \tag{8}$$

where

$$D = \mathrm{Pe}^e_1 e^{-4G^e_{32}\mathrm{Pe}^{e,\mathbf{m}}_1} + \mathrm{Pe}^e_2 e^{-4G^e_{13}\mathrm{Pe}^{e,\mathbf{m}}_2} + \mathrm{Pe}^e_3 e^{-4G^e_{21}\mathrm{Pe}^{e,\mathbf{m}}_3}$$

and the non-dimensional parameters G^e_{ij}, Pe^e_i and $\mathrm{Pe}^{e,\mathbf{m}}_i$ are defined in the sequel.
The geometrical factors G^e_{ij} are defined by

$$G^e_{ij} = A^e \boldsymbol{\nabla}\phi^e_i \cdot \boldsymbol{\nabla}\phi^e_j$$

where A^e is the area of the element Ω^e.
Firstly, the geometrical matrix $[G^e_{ij}]$ has the following properties:

1) it is symmetric,

2) the sum of the elements of a row or a column is zero,

3) its adjoint matrix $\det(G^e_{ij})\,\big[G^e_{ij}\big]$ is proportional to a matrix whose elements are the same.

Therefore this matrix, which characterizes the shape of element e, has only two degrees of freedom.
Secondly, the element Péclet numbers Pe^e_i are defined by

$$\mathrm{Pe}^e_i = \frac{A^e}{\kappa}\mathbf{v}\cdot\boldsymbol{\nabla}\phi^e_i$$

while the element Péclet numbers associated to direction $\mathbf{m}$ are defined by

$$\mathrm{Pe}^{e,\mathbf{m}}_j = \frac{A^e(\mathbf{m}\cdot\mathbf{v})(\mathbf{m}\cdot\boldsymbol{\nabla}\phi^e_j)}{\kappa(\mathbf{m}\cdot\mathbf{m})}$$

and coincide with Pe^e_i when $\mathbf{m} = \mathbf{v}$. It is easy to show that these Péclet numbers satisfy the relations

$$\sum_{n(e)} \mathrm{Pe}^{e,\mathbf{m}}_n = 0$$

$$\sum_{e(i)} \mathrm{Pe}^{e,\mathbf{m}}_i = 0$$

where $n(e)$ and $e(i)$ stand for the nodes belonging to element Ω^e, and the elements sharing node i, respectively.

III. First results

We here present some typical results obtained by means of our method in comparison with results obtained with the classical SUPG method [BRO 82]. As already mentioned, the SUPG method achieves stabilization by adding an upwinding term to the original weak formulation. The upwinding level is controlled by a specific element Péclet number $\mathrm{Pe}^e = \|\mathbf{v}\| h^e/\kappa$, where h^e stands for the element characteristic size.

A. *Thermal boundary layer problem*

This test has been proposed in [FRA 92]. The problem statement is depicted in Fig. 4.

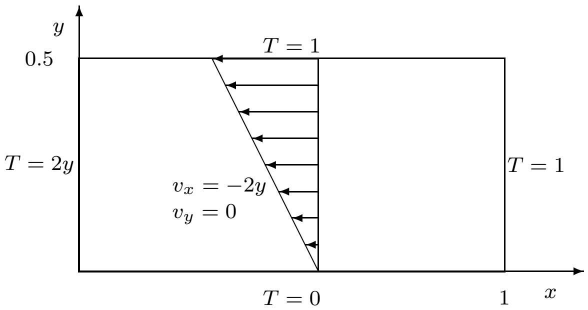

Fig. 4. Thermal boundary layer problem: problem statement

This situation may be viewed as the modelling of the formation of a thermal boundary layer at the lower and downstream boundaries of a fully developed shear flow between two parallel plates, where the upper plate is moving while the bottom plate is fixed. The mesh consists of rectangular triangles, providing 21 equally spaced nodes in the x-direction and 11 equally spaced nodes in the y-direction.
A first numerical experiment has been performed with a global Péclet number of 10^3. The resulting element Péclet number is $\mathrm{Pe}^e = 50y$, with a maximum of 25 along the upper plate (h^e being taken as the horizontal element length). From a numerical viewpoint, the problem is diffusive in the vicinity of the bottom plate while it is more and more advective for increasing y. Figs. 5 and 6 show the temperature isolines obtained by means of the SUPG method and our method, respectively. It can be observed that both solutions are numerically stable, but that the SUPG solution suffers from a slight overshoot contrarily to the solution provided by our exact Petrov-Galerkin FE method.
A second numerical experiment has been performed with a global Péclet number of 10^6. From a numerical viewpoint, the problem is now strongly advective. Figs. 7 and 8 show the temperature isolines obtained with the SUPG method and our method, respectively. The solution provided by our method clearly behaves much better than the SUPG solution. In particular, it is noticeable that our solution accomodates a much smaller overshoot.

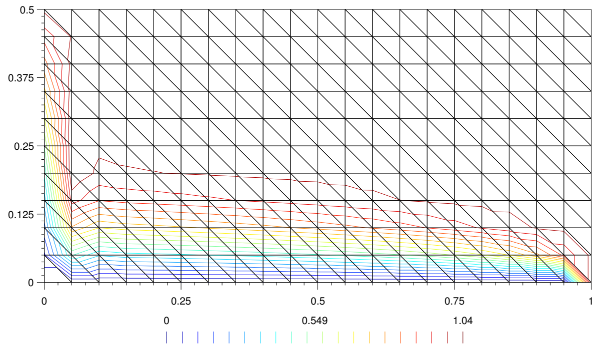

Fig. 5. Thermal boundary layer problem: results from SUPG method with Pe $=$ 1 000. An overshoot of 4% is observed near the outflow ($x = 0.05$, $y = 0.25$).

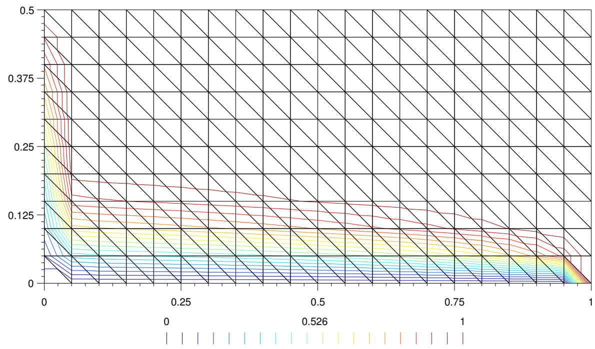

Fig. 6. Thermal boundary layer problem: results from our exact Petrov-Galerkin FE method with Pe $=$ 1 000 and $\mathbf{m} = \mathbf{v}$. No overshoot is observed.

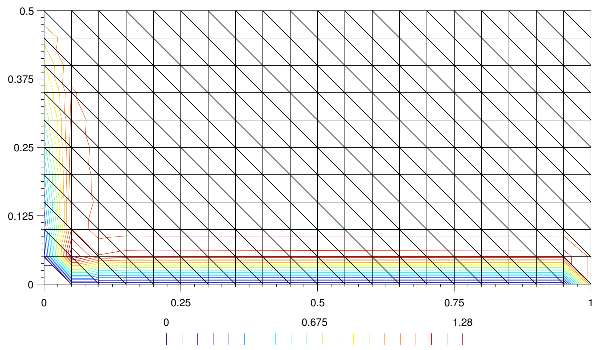

Fig. 7. Thermal boundary layer problem: results for the SUPG method with Pe $=$ 1 000 000. An overshoot of 28% is observed near the outflow ($x = 0.05$, $y = 0.05$).

B. *Brezzi's problem*

This second test, here named "Brezzi's problem", is suggested in [BRE 98]. The problem statement with the detailed boundary conditions is depicted in Fig. 9. The velocity field is that of a rigid body uniformly rotating around the origin.

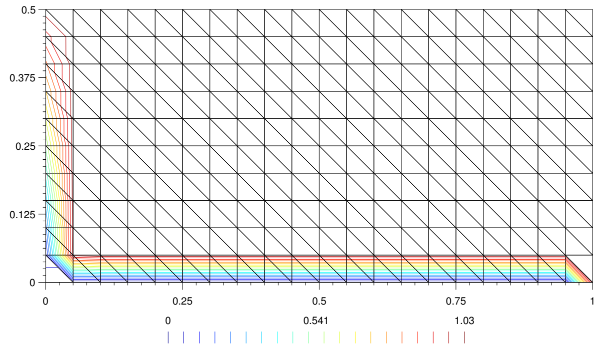

Fig. 8. Thermal boundary layer problem: results for our method with Pe $=$ 1 000 000 and $\mathbf{m} = \mathbf{v}$. An overshoot of 3% is observed along the bottom boundary layer ($y = 0.05$).

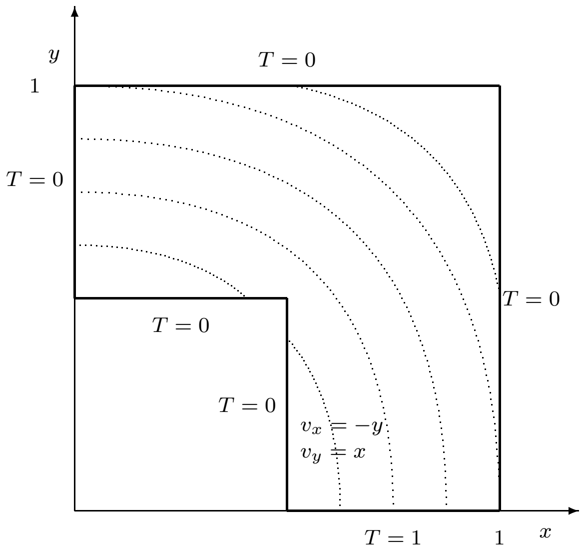

Fig. 9. Brezzi's problem: problem statement (dotted lines indicate the flow streamlines).

In the first numerical experiment, the global Péclet number was about 200. We have used the mesh shown in Fig. 10, for which the problem is partly diffusion-dominated and partly convection-dominated. This test is severe in the sense that the solution presents a first outflow boundary layer, together with a second boundary layer which terminates in the vicinity of the reentrant corner. Figs. 11 and 12 show the temperature isolines obtained with the SUPG method and our method, respectively. Again, the solution obtained by means of our method is of good quality and does not suffer from any overshoot.

Other numerical experiments with much higher Péclet numbers were also performed. The results obtained from our method were still very good and even better than those obtained with SUPG method (Figs. 13 and 14).

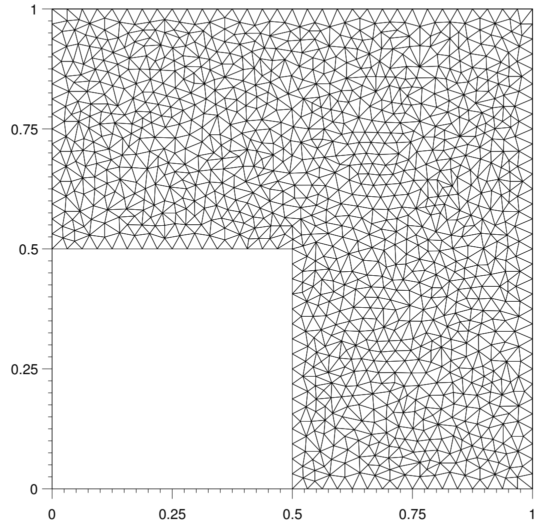

Fig. 10. Brezzi's problem: mesh (1219 nodes, 2308 elements).

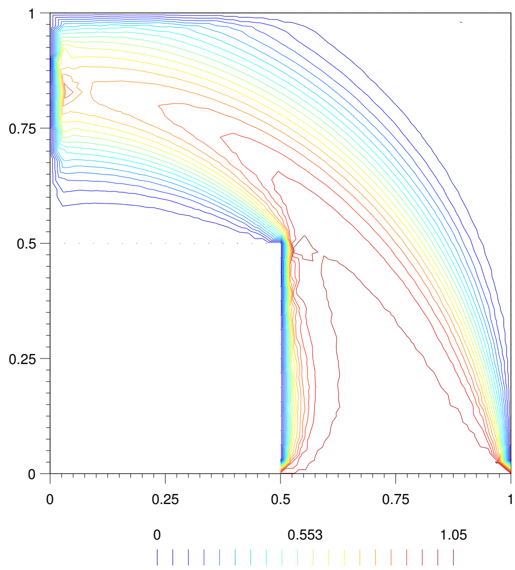

Fig. 11. Brezzi's problem: results from the SUPG method with Pe=200. Overshoot: 5%.

IV. Discussion and Conclusions

An exact Petrov-Galerkin FE method to solve convection-dominated problems has been proposed. Our approach is based on using test functions that provide exact nodal values for a selected class of 1D solutions. Results of very high quality have been obtained so far. In order to build a general discretization and solution algorithm, the research effort will be pursued focusing on the following issues:

1) To determine an optimal balance between the test func-

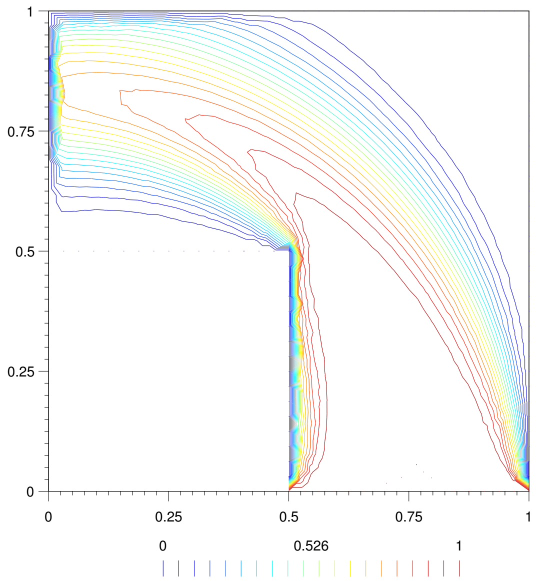

Fig. 12. Brezzi's problem: results from our method with Pe=200 and $\mathbf{m} = \mathbf{v}$ inside the domain + special treatment at the boundary. No overshoot.

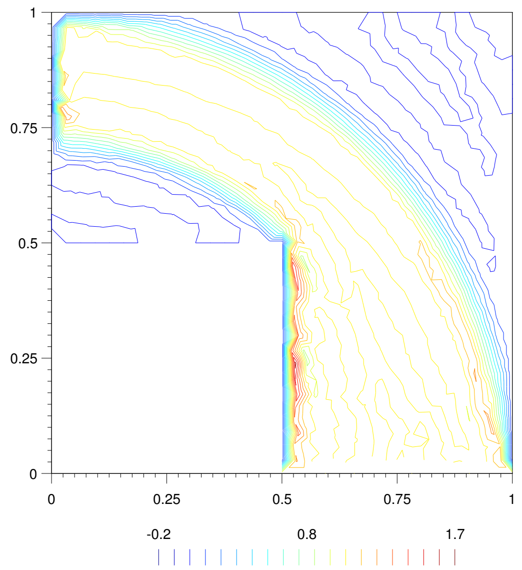

Fig. 13. Brezzi's problem: results from the SUPG method with $\mathrm{Pe} = 1\,000\,000$. Overshoot: 64%, undershoot: -5%.

tions provided by our exact Petrov-Galerkin method and the classical linear test functions of the Galerkin method. This balance is required to address the problem coming from possibly very small denominator D in Eqs. 6-7-8 in case of diffusion-dominated situations, thereby providing unacceptably large test functions.

2) To select the numerical direction $\mathbf{m}$ according to an appropriate criterion.

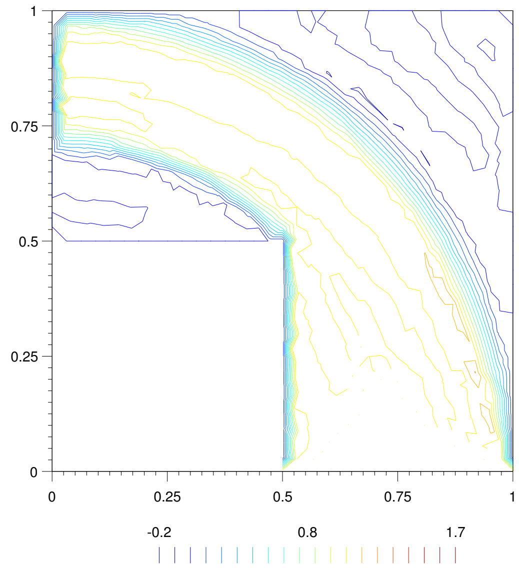

Fig. 14. Brezzi's problem: results from our method with $\mathrm{Pe} = 1\,000\,000$ and $\mathbf{m} = \mathbf{v}$ inside the domain + special treatment at the boundary. Overshoot: 21%, undershoot: -12% . Same scale as in Fig. 13.

3) To extend the method to problems with flux (or natural) boundary conditions, to problems with source terms at the right-hand side (possibly resulting from transient effects), and to 3D problems.

4) To use higher order elements.

Although definite conclusions cannot be drawn before having addressed the above issues, it is worth nothing that our research clearly indicates that accurate and reliable exact Petrov-Galerkin FE methods can be built. In addition, the method we propose directly leads to defining appropriate dimensionless element Péclet numbers and geometrical factors in order to exactly characterize the local transport intensity and orientation with respect to the element shape, hence providing rigourous tools to solve the advection-diffusion problem on general unstructured meshes.

REFERENCES

[BAI 93] BAIOCCHI C., BREZZI F., FRANCA L., *Virtual bubbles and Galerkin-least-squares type methods*, vol. 105, p. 125–141, Computer Methods in Applied Mechanics and Engineering, 1993.

[BRE 98] BREZZI F., MARINI D., RUSSO A., *Applications of the pseudo residual-free bubbles to the stabilization of convection-diffusion problems*, vol. 166, p. 51–63, Computer Methods in Applied Mechanics and Engineering, 1998.

[BRO 82] BROOKS A., HUGHES T., *Streamline Upwind / Petrov Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations*, vol. 32, p. 199–259, Computer Methods in Applied Mechanics and Engineering, 1982.

[COC 00] COCKBURN B., KARNIADAKIS G., SHU C.-W., *Discontinuous Galerkin Methods. Theory, Computation and Applications.*, Notes in Computational Science and Engineering, Springer, 2000.

[FRA 92] FRANCA L., FREY S., HUGHES T., *Stabilized finite element methods: I. Application to the advective diffusive model*, vol. 95, p. 253–276, Computer Methods in Applied Mechanics and Engineering, 1992.

[HUG 79] HUGHES T., BROOKS A., *A multidimensional upwind scheme with no crosswind diffusion*, vol. 34, p. 19–35, T.J.R. Hughes (Ed), Finite Element Methods for Convection Dominated Flows, ASME New-York, 1979.

[HUG 89] HUGHES T., FRANCA L., HULBERT G., *A new finite element formulation for computational fluid dynamics: VIII. The galerkin/least-squares method for advective-diffusive equations*, vol. 73, p. 173–189, Computer Methods in Applied Mechanics and Engineering, 1989.

[HUG 00] HUGHES T., ENGEL G., MAZZEI L., LARSON M., *The continuous Galerkin method is conservative*, vol. 163, p. 467–488, Journal of Computational Physics, 2000.

[JOH 84] JOHNSON C., NAVERT U., PITKARANTA J., *Finite element methods for linear hyperbolic problems*, vol. 45, p. 285–312, Computer Methods in Applied Mechanics and Engineering, april 1984.

[NES 03] NESLITURK A., HARARI I., *The nearly-optimal Petrov-Galerkin method for convection-diffusion problems*, vol. 192, p. 2501–2519, Computer Methods in Applied Mechanics and Engineering, 2003.

[PER 96] PERELLA A., *A class of Petrov-Galerkin finite element methods for the numerical solution of the stationary convection-diffusion equation*, Thesis, 1996.